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Fractional Differentiation of the Commutator of the Hilbert Transform*

C. SEGOVIA

Princeton University, Princeton, New Jersey

AND

R. L. WHEEDEN

Rutgers University, New Brunswick, New Jersey

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Letting H denote the Hilbert transform operator, we consider the commutator

$$(aH - Ha)f(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{x - y} f(y)\, dy,$$

in the case that f belongs to L^p, $1 < p < \infty$, and a belongs to the Lebesgue (Sobolev) space L_α^∞, $0 < \alpha < 1$. Our principal result is an analogue for such α of a theorem of A. P. Calderón [1] corresponding essentially to the case $\alpha = 1$.

INTRODUCTION

In this paper we will derive a result related to a theorem of A. P. Calderón on the commutator of the Hilbert transform. (See [1].) If

$$(Hf)(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f(y)}{x - y}\, dy$$

denotes the Hilbert transform of f and

$$(Af)(x) = a(x) f(x),$$

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we form the commutator

$$(AH - HA)(f)(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{x - y} f(y)\, dy.$$

Let L_α^p, $1 \leqslant p \leqslant \infty$, $\alpha > 0$, denote the class of functions which are Bessel potentials of order α of L^p functions, and let $\| \cdot \|_{p,\alpha}$ denote the norm in L_α^p. The usual L^p norm is $\| \cdot \|_p$.

Our main result is the following theorem.

THEOREM 1. *Let $f \in L^p$, $1 < p < \infty$, and $a \in L_\alpha^\infty$, $0 < \alpha < 1$. Then the commutator $(AH - HA) f \in L_\alpha^p$. Moreover,*

$$\|(AH - HA) f \|_{p,\alpha} \leqslant C \| f \|_p \| a \|_{\infty,\alpha} ,$$

where the constant C is independent of a and f.

One may think of Theorem 1 as an analogue for differentiation of a fractional order of the one-dimensional version of Calderón's result in [1]. (For $\alpha = 1$, however, the hypothesis in [1] is that a belong to Lip 1, not L_1^∞.) Our proof for $0 < \alpha < 1$ is similar in several respects to that given in [1] for $\alpha = 1$—in particular, it uses complex variable methods and an "area" integral. The added complication in the proof is due to difficulties which arise from taking a derivative of fractional order.

Theorem 1 has analogues for $a \in L_\alpha^r$, $r < \infty$. In particular,

THEOREM 2. *Let $f \in L^p$, $1 < p < \infty$, and $a \in L_\alpha^r$, $1 < r < \infty$, $0 < \alpha < 1$. Then the commutator $(AH - HA) f \in L_\alpha^q$ for $(1/q) = (1/p) + (1/r)$ and $1 < q < \infty$. Moreover,*

$$\|(AH - HA) f \|_{q,\alpha} \leqslant C \| f \|_p \| a \|_{r,\alpha} ,$$

where C is independent of a and f.

Theorem 2 follows from the results of [1] by interpolation. To see this, one fixes f and considers $(AH - HA) f$ as a linear operator on a. The result then follows from applying Theorem 10 of [3] to an interpolation between the extreme cases $a \in L^r$ and $a \in L_1^r$.

Theorems 1 and 2 have analogues for $\alpha > 1$. When $1 < \alpha < 2$ for example, they are true as stated, and the proofs are similar to those for $0 < \alpha < 1$ but require a few modifications. The case $\alpha = 2$ is treated in [2, Theorem (5.2)]. For simplicity we shall consider from now on only the case $0 < \alpha < 1$, $r = \infty$.

PRELIMINARIES

Aside from well-known results about Hilbert transforms, H^p spaces, etc., we shall need three basic facts in order to prove Theorem 1. We list these below as lemmas.

Recall that if $F \in L_\alpha^p$, $1 \leqslant p \leqslant \infty$, $0 < \alpha < 1$, then $F = \psi * G_\alpha$, where $\psi \in L^p$ and $\| F \|_{p,\alpha} = \| \psi \|_p$. Here $G_\alpha \geqslant 0$, $G_\alpha \in L^1$, $\hat{G}_\alpha(x) = (1 + | x |^2)^{-\alpha/2}$ and

$$|(d^k/dx^k)\, G_\alpha(x)| \leqslant c_{\alpha,k}\, | x |^{\alpha-k-1}$$

for $k = 0, 1, 2,\ldots$ (see, for example, [5, p. 192]).

LEMMA 1. *For $1 \leqslant p \leqslant \infty$ and $0 < \alpha < 1$, consider the hyper-singular integral*

$$\tilde{F}_\epsilon(x) = \int_{|h| > \epsilon} \frac{F(x+h) - F(x)}{| h |^{1+\alpha}}\, dh.$$

If $F \in L^p$ and $\| \tilde{F}_\epsilon \|_p \leqslant M < \infty$ for all $\epsilon > 0$, then $F \in L_\alpha^p$ and

$$\| F \|_{p,\alpha} \leqslant C(\| F \|_p + M).$$

This lemma is a variant of Theorem 2 of [7], or Theorem 1 of [9]. The proof follows from a very minor modification of the argument on p. 432 of [9].

LEMMA 2. *If $a = \psi * G_\alpha$, $0 < \alpha < 1$, then*

$$| a(x) - a(y)| \leqslant C \| \psi \|_\infty\, | x - y |^\alpha.$$

(See, for example, [3].)

Let $h(x, y) = h(x + iy)$ belong to the Hardy class H^p, $p > 0$; that is, let h be analytic for $y > 0$ and satisfy

$$\int_{-\infty}^{\infty} | h(x + iy)|^p\, dx \leqslant C^p$$

for every $y > 0$ and some $p > 0$. For $0 < \alpha < 1$, let

$$h^{(\alpha)}(x, y) = \int_0^\infty \frac{\partial h}{\partial y}(x, y + s)\, s^{-\alpha}\, ds$$

be the α-th derivative of h and

$$S_\alpha(h)(x;\gamma) = \left(\iint_{|x-t|<\gamma y} y^{2(\alpha-1)} \mid h^{(\alpha)}(t,y)|^2 \, dt \, dy\right)^{1/2}$$

be the area integral of h or order α. (See [6].)

LEMMA 3. *Let $h(x, y)$, $y > 0$, belong to H^p for some $p > 0$ and let $h(x) = \lim_{y\to 0} h(x, y)$. There exist positive constants c_1 and c_2 independent of h such that*

$$c_1 \| h(x)\|_p \leqslant \| S_\alpha(h)(x;\gamma)\|_p \leqslant c_2 \|h(x)\|_p .$$

For a proof, see Lemma 7 of [6].

We shall prove Theorem 1 by showing that the function $F = (AH - HA)f$ satisfies the hypotheses of Lemma 1. This will be accomplished in the following stages. In Section 1, we compute the hypersingular integral of Lemma 1 for $(AH - HA)f$, obtaining a sum of four integrals. In Section 2, we use Lemma 2 to show that the L^q norms of two of these integrals behave. In Section 3, we apply complex variable methods and Lemma 3 to estimate the L^p norm of the remaining integrals. In these sections we will assume that a and f are smooth. Section 4 contains an approximation argument to remove the unnecessary restrictions on a and f.

SECTION 1

Let us assume in this and the following two sections that f is infinitely differentiable with compact support, $\psi \in L^1 \cap L^\infty$ and $a = \psi * G_\alpha$, $0 < \alpha < 1$. If $F = (AH - HA)f$ then

$$\tilde{F}_\epsilon(x) = \int_{|h|>\epsilon} \left\{ \text{p.v.} \int_{-\infty}^{\infty} \frac{a(x+h) - a(y)}{x+h-y} f(y)\, dy - \text{p.v.} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{x-y} f(y)\, dy \right\} \frac{dh}{|h|^{1+\alpha}} .$$

Writing $a(x+h) - a(y) = [a(x+h) - a(x)] + [a(x) - a(y)]$, we get

$$\tilde{F}_\epsilon(x) = \int_{|h|>\epsilon} \frac{a(x+h) - a(x)}{|h|^{1+\alpha}} (Hf)(x+h)\, dh + I_\epsilon(x), \tag{1.1}$$

where

$$I_\epsilon(x) = \int_{|h|>\epsilon} \Big\{ \text{p.v.} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{x + h - y} f(y)\, dy - \text{p.v.} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{x - y} f(y)\, dy \Big\} \frac{dh}{|h|^{1+\alpha}} .$$

We wish to simplify this expression. It follows from Lemma 2 that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{x - y} f(y)\, dy$$

converges absolutely—in fact,

$$\int_{-\infty}^{\infty} \left| \frac{a(x) - a(y)}{x - y} \right| |f(y)|\, dy \leqslant C \int_{-\infty}^{\infty} \frac{|f(y)|}{|x - y|^{1-\alpha}}\, dy$$
$$\leqslant C \int_{|x-y|<1} |x - y|^{\alpha-1}\, dy + c \int_{|x-y|>1} |f(y)|\, dy$$

is bounded. Thus

$$I_\epsilon(x) = \int_{|h|>\epsilon} \Big\{ \lim_{\eta\to 0} \int_{|x+h-y|>\eta} [a(x) - a(y)] \Big[\frac{1}{x + h - y} - \frac{1}{x - y} \Big] f(y)\, dy \Big\} \times \frac{dh}{|h|^{1+\alpha}} .$$

We want to interchange the integration in h with the limit in η. As just remarked, the part

$$\int_{|x+h-y|>\eta} \frac{a(x) - a(y)}{x - y} f(y)\, dy$$

is bounded in x, h and η. Moreover, if g^* denotes the maximal Hilbert transform of g,

$$g^*(x) = \sup_{\eta>0} \left| \int_{|x-y|>\eta} \frac{g(y)}{x - y}\, dy \right| ,$$

then the part

$$\left| \int_{|x+h-y|>\eta} [a(x) - a(y)] \frac{f(y)}{x + h - y}\, dy \right|$$
$$\leqslant |a(x)| f^*(x + h) + (af)^* (x + h).$$

Since a is bounded, f^* and $(af)^*$ belong to L^2 (say), and since $|h|^{-1-\alpha}$ is integrable over $|h| > \epsilon$, we have from Young's convolution theorem and the Lebesgue dominated convergence theorem that

$$I_\epsilon(x) = \lim_{\eta\to 0} \int_{|h|>\epsilon} \left\{ \int_{|x+h-y|>\eta} [a(x) - a(y)] \frac{-h}{(x-y)(x+h-y)} f(y)\, dy \right\} \times \frac{dh}{|h|^{1+\alpha}}.$$

We now want to interchange the integrations in h and y. To accomplish this, we write

$$\int \left| \frac{a(x)-a(y)}{x-y} \right| |f(y)|\, dy \int_{\substack{|h|>\epsilon \\ |x+h-y|>\eta}} \frac{|h|\, dh}{|h|^{1+\alpha} |x+h-y|}$$
$$= \int \frac{|a(x)-a(y)|}{|x-y|^{1+\alpha}} |f(y)|\, dy \int_{\substack{|h|>\epsilon/|x-y| \\ |1+h|>\eta/|x-y|}} \frac{dh}{|h|^\alpha |1+h|}$$

If $\epsilon/|x-y| > 2$, the inner integral is at most

$$2 \int_{|h|>\epsilon/|x-y|} \frac{dh}{|h|^{1+\alpha}} = C_\epsilon |x-y|^\alpha.$$

However,

$$\int \frac{|a(x)-a(y)|}{|x-y|^{1+\alpha}} |x-y|^\alpha |f(y)|\, dy < \infty.$$

If, on the other hand, $\epsilon/|x-y| < 2$ then the inner integral is at most

$$\int_{\substack{2>|h|>\epsilon/|x-y| \\ |1+h|>\eta/|x-y|}} \frac{dh}{|h|^\alpha |1+h|} + 2 \int_{|h|>2} \frac{dh}{|h|^{1+\alpha}}$$
$$\leqslant A_\eta \left[|x-y| \int_{2>|h|>\epsilon/|x-y|} \frac{dh}{|h|^\alpha} + 1 \right]$$
$$\leqslant A_{\epsilon,\eta} [|x-y|^\alpha + |x-y| + 1].$$

However,

$$\int_{|x-y|>\epsilon/2} \frac{|a(x)-a(y)|}{|x-y|^{1+\alpha}} (|x-y|^\alpha + |x-y| + 1) |f(y)|\, dy < \infty.$$

Hence,

$$I_\epsilon(x) = \lim_{\eta\to 0} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{x - y} f(y)\, dy \int_{\substack{|h|>\epsilon \\ |x-y+h|>\eta}} \frac{-h dh}{|h|^{1+\alpha}(x - y + h)},$$

or

$$I_\epsilon(x) = \lim_{\eta\to 0} \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{|x - y|^{1+\alpha}} f(y) \operatorname{sign}(x - y)\, b_{\epsilon,\eta}(x - y)\, dy, \quad (1.2)$$

where

$$b_{\epsilon,\eta}(z) = \int_{\substack{|h|>\epsilon/|z| \\ |1-h|>\eta/|z|}} \frac{h dh}{|h|^{1+\alpha}(1 - h)}. \quad (1.3)$$

LEMMA 4. *For* $0 < \alpha < 1$,

$$b = \text{p.v.} \int_{-\infty}^{\infty} \frac{h dh}{|h|^{1+\alpha}(1 - h)} = \lim_{\delta\to 0} \int_{|1-h|>\delta} = -\pi \tan \frac{\pi}{2} (1 - \alpha).$$

Proof. The precise value of b will be important in Section 3. We sketch a proof of Lemma 4 which uses complex integration.

$$b = \text{p.v.} \int_0^{\infty} \left(\frac{1}{1 - h} - \frac{1}{1 + h}\right) \frac{dh}{h^\alpha} = 2\text{p.v.} \int_0^{\infty} \frac{h^{1-\alpha}}{1 - h^2}\, dh.$$

Choosing $z^{1-\alpha}$ analytic for $I(z) \geqslant 0$ and real on the positive real axis, and performing a contour integration over the boundary of the domain $R(z) \geqslant 0$, $I(z) \geqslant 0$, $|z - 1| \geqslant \delta$, $|z| \leqslant R$, we obtain

$$2\text{p.v.} \int_0^{\infty} \frac{h^{1-\alpha}}{1 - h^2}\, dh = -i\pi + 2ie^{i(\pi/2)(1-\alpha)} \int_0^{\infty} \frac{h^{1-\alpha}}{1 + h^2}\, dh.$$

By the theory of residues and an integration over the boundary of the domain $I(z) \geqslant 0$, $|z| \leqslant R$, we obtain

$$\int_0^{\infty} \frac{h^{1-\alpha}}{1 + h^2}\, dh = \frac{\pi}{2 \cos(\pi/2)(1 - \alpha)},$$

and the lemma follows.

Combining (1.1) and (1.2), we obtain

$$\begin{aligned}
\tilde{F}_\epsilon(x) &= -\int_{|x-y|>\epsilon} \frac{a(x)-a(y)}{|x-y|^{1+\alpha}} (Hf)(y)\,dy \\
&\quad + b\int_{|x-y|>\epsilon} \frac{a(x)-a(y)}{|x-y|^{1+\alpha}} \operatorname{sign}(x-y) f(y)\,dy \\
&\quad + \lim_{\eta\to 0}\int_{|x-y|>\epsilon} \frac{a(x)-a(y)}{|x-y|^{1+\alpha}} \operatorname{sign}(x-y)[b_{\epsilon,\eta}(x-y)-b] f(y)\,dy \\
&\quad + \lim_{\eta\to 0}\int_{|x-y|<\epsilon} \frac{a(x)-a(y)}{|x-y|^{1+\alpha}} \operatorname{sign}(x-y)\, b_{\epsilon,\eta}(x-y) f(y)\,dy \\
&= I_1 + I_2 + I_3 + I_4 . \qquad (1.4)
\end{aligned}$$

These are the four integral expressions referred to earlier. They of course depend on ϵ and x.

Section 2

In this section we will show that the L^p norms of I_3 and I_4 are bounded uniformly in ϵ by a constant times $\| a \|_{\infty,\alpha} \| f \|_p$. The burden of the proof is verifying the following lemma.

Lemma 5. *For* $0 < \eta < \epsilon$, $0 < \alpha < 1$,

(i) $$| b_{\epsilon,\eta}(z) - b | \leqslant C \left[\left(\frac{\epsilon}{|z|} \right)^{1-\alpha} + \log^+ \left(2 - \frac{2\epsilon}{|z|} \right)^{-1} \right]$$

for $|z| > \epsilon$, *and*

(ii) $$| b_{\epsilon,\eta}(z)| \leqslant C \left[\left(\frac{|z|}{\epsilon} \right)^{\alpha} + \log^+ \left(\frac{\epsilon}{|z|} - 1 \right)^{-1} \right]$$

for $|z| < \epsilon$, *where the constant* C *is independent of* ϵ *and* η.

We begin by proving (ii). The part of

$$b_{\epsilon,\eta}(z) = \int_{\substack{|h|>\epsilon/|z| \\ |1-h|>\eta/|z|}} \frac{hdh}{(1-h)|h|^{1+\alpha}}$$

where the integration is extended only over $h < 0$ is

$$-\int_{\substack{h>\epsilon/|z| \\ 1+h>\eta/|z|}} \frac{dh}{(1+h)\,h^\alpha} = -\int_{\epsilon/|z|}^{\infty} \frac{dh}{(1+h)\,h^\alpha}$$

since $\epsilon > \eta$. In absolute value this at most a constant times

$$\int_{\epsilon/|z|}^{\infty} \frac{dh}{h^{\alpha+1}} = 0\left(\frac{|z|}{\epsilon}\right)^{\alpha}.$$

Since $\epsilon/|z| > 1$, the part of $b_{\epsilon,\eta}(z)$ where the integration is extended over $h > 0$ is

$$\int_{\substack{h>\epsilon/|z| \\ h>1+(\eta/|z|)}} \frac{dh}{(1-h)\,h^{\alpha}}\,.$$

The integrand has constant sign so since we are interested in absolute values we drop the condition $h > 1 + (\eta/|z|)$. If $\epsilon/|z| > 2$ then $h - 1 > h/2$ and we obtain at most

$$2\int_{\epsilon/|z|}^{\infty} \frac{dh}{h^{\alpha+1}} = 0\left(\frac{|z|}{\epsilon}\right)^{\alpha}.$$

If $1 < \epsilon/|z| < 2$, write

$$\begin{aligned}
&\int_{\epsilon/|z|}^{\infty} \frac{dh}{(h-1)\,h^{\alpha}} \\
&\quad = \int_{\epsilon/|z|}^{2} + \int_{2}^{\infty} \leqslant \int_{\epsilon/|z|}^{2} \frac{dh}{h-1} + 2\int_{2}^{\infty} \frac{dh}{h^{\alpha+1}} \\
&\quad = -\log\left(\frac{\epsilon}{|z|} - 1\right) + 0(1) = \log^{+}\left(\frac{\epsilon}{|z|} - 1\right)^{-1} + 0(1) \\
&\quad \leqslant \log^{+}\left(\frac{\epsilon}{|z|} - 1\right)^{-1} + 0\left(\frac{|z|}{\epsilon}\right)^{\alpha}.
\end{aligned}$$

This completes the proof of (ii).

The proof of (i) is similar but more involved. Now $\epsilon/|z| < 1$ and

$$b_{\epsilon,\eta}(z) - b = \int_{\substack{|h|>\epsilon/|z| \\ (1-h)>\eta/|z|}} \frac{hdh}{(1-h)|\,h\,|^{1+\alpha}} - \text{p.v.} \int_{-\infty}^{\infty} \frac{hdh}{(1-h)|\,h\,|^{1+\alpha}}\,.$$

Consider first the parts where the integrations are extended over $h < 0$. These give

$$-\int_{\substack{h>\epsilon/|z| \\ 1+h>\eta/|z|}} \frac{dh}{(1+h)\,h^{\alpha}} + \int_{0}^{\infty} \frac{dh}{(1+h)\,h^{\alpha}} = \int_{0}^{\epsilon/|z|} \frac{dh}{(1+h)\,h^{\alpha}}\,,$$

since the condition $1 + h > \eta/|z|$ in the first integral is redundant ($\epsilon > \eta$). The last integral is less than

$$\int_0^{\epsilon/|z|} \frac{dh}{h^\alpha} = 0\left(\frac{\epsilon}{|z|}\right)^{1-\alpha}.$$

Consider next the parts of $b_{\epsilon,\eta} - b$ with integrations over $h > 0$. We will first estimate the portion

$$\int_0^{\epsilon/|z|} \frac{dh}{(1-h)\,h^\alpha} \qquad \left(\frac{\epsilon}{|z|} < 1\right)$$

of b. If $\epsilon/|z| < (1/2)$ then $1 - h > (1/2)$, so we obtain at most

$$2\int_0^{\epsilon/|z|} \frac{dh}{h^\alpha} = 0\left(\frac{\epsilon}{|z|}\right)^{1-\alpha}.$$

If, on the contrary, $1/2 < \epsilon/|z| < 1$, then

$$\begin{aligned}
&\int_0^{\epsilon/|z|} \frac{dh}{(1-h)\,h^\alpha} \\
&= \int_0^{1/2} + \int_{1/2}^{\epsilon/|z|} \leqslant C\int_0^{1/2} \frac{dh}{h^\alpha} + C\int_{1/2}^{\epsilon/|z|} \frac{dh}{1-h} \\
&= \log\left(2 - \frac{2\epsilon}{|z|}\right)^{-1} + 0(1) \leqslant \log^+\left(2 - \frac{2\epsilon}{|z|}\right)^{-1} + 0\left(\frac{\epsilon}{|z|}\right)^{1-\alpha}.
\end{aligned}$$

The remaining parts of $b_{\epsilon,\eta} - b$ with integrations over $h > 0$ give

$$\int_{\substack{h > \epsilon/|z| \\ |1-h| > \eta/|z|}} \frac{dh}{(1-h)\,h^\alpha} - \text{p.v.} \int_{\epsilon/|z|}^{\infty} \frac{dh}{(1-h)\,h^\alpha}. \tag{2.1}$$

Here $\epsilon/|z| < 1$ and $\eta < \epsilon$ imply $1 - \eta/|z| > 0$. There are two possibilities: either $\epsilon/|z| < 1 - \eta/|z|$ or $1 - \eta/|z| < \epsilon/|z|$. In the first case, (2.1) is

$$-\text{p.v.} \int_{1-\eta/|z|}^{1+\eta/|z|} \frac{dh}{(1-h)\,h^\alpha} = \lim_{\delta\to 0} \int_\delta^{\eta/|z|} \left\{\frac{1}{(1+h)^\alpha} - \frac{1}{(1-h)^\alpha}\right\} \frac{dh}{h}.$$

Since the expression in curly brackets is always negative, we may consider

$$\int_0^{\eta/|z|} \left\{\frac{1}{(1-h)^\alpha} - \frac{1}{(1+h)^\alpha}\right\} \frac{dh}{h} \leqslant \int_0^{\epsilon/|z|} \left\{\frac{1}{(1-h)^\alpha} - \frac{1}{(1+h)^\alpha}\right\} \frac{dh}{h}.$$

If $(\epsilon/|z|) < (1/2)$, it follows from the mean-value theorem that the expression in curly brackets is $O(h)$, and we obtain

$$\int_0^{\epsilon/|z|} O(h)\,\frac{dh}{h} = O\left(\frac{\epsilon}{|z|}\right) = O\left(\frac{\epsilon}{|z|}\right)^{1-\alpha}.$$

If $(1/2) < (\epsilon/|z|) < 1$,

$$\int_0^{1/2} + \int_{1/2}^{\epsilon/|z|} \left\{\frac{1}{(1-h)^\alpha} - \frac{1}{(1+h)^\alpha}\right\} \frac{dh}{h}$$
$$\leqslant \int_0^{1/2} O(h)\,\frac{dh}{h} + \int_{1/2}^{1} \frac{1}{(1-h)^\alpha}\,\frac{dh}{h} + \int_{1/2}^{1} \frac{1}{(1+h)^\alpha}\,\frac{dh}{h}$$
$$= O(1) = O\left(\frac{\epsilon}{|z|}\right)^{1-\alpha},$$

which completes the proof for the first possibility.

In the case $1 - (\eta/|z|) < (\epsilon/|z|)$, (2.1) is

$$-\text{p.v.} \int_{\epsilon/|z|}^{1+\eta/|z|} \frac{dh}{(1-h)\,h^\alpha} = \lim_{\delta\to 0} \left[\int_\delta^{\eta/|z|} \frac{dh}{h(1+h)^\alpha} - \int_\delta^{1-\epsilon/|z|} \frac{dh}{h(1-h)^\alpha}\right].$$

Since $1 - \epsilon/|z| < \eta/|z|$ in this case, we obtain

$$-\int_0^{1-\epsilon/|z|} \left\{\frac{1}{(1-h)^\alpha} - \frac{1}{(1+h)^\alpha}\right\} \frac{dh}{h} + \int_{1-\epsilon/|z|}^{\eta/|z|} \frac{dh}{h(1+h)^\alpha}.$$

Note that $\epsilon < |z| < \epsilon + \eta < 2\epsilon$, so that $(1/2) < (\epsilon/|z|) < 1$. Therefore,

$$\int_{1-\epsilon/|z|}^{\eta/|z|} \frac{dh}{h(1+h)^\alpha}$$
$$\leqslant \int_{1-\epsilon/|z|}^{1} \frac{dh}{h} = -\log\left(1 - \frac{\epsilon}{|z|}\right)$$
$$= \log\left(2 - \frac{2\epsilon}{|z|}\right)^{-1} + \log 2 \leqslant \log^+\left(2 - \frac{2\epsilon}{|z|}\right)^{-1} + O\left(\frac{\epsilon}{|z|}\right)^{1-\alpha}.$$

Finally, since $0 < 1 - (\epsilon/|z|) < (1/2)$, the mean-value theorem shows that

$$\int_0^{1-\epsilon/|z|} \left\{\frac{1}{(1-h)^\alpha} - \frac{1}{(1+h)^\alpha}\right\} \frac{dh}{h} \leqslant \int_0^{1/2} O(h)\,\frac{dh}{h} = O\left(\frac{\epsilon}{|z|}\right)^{1-\alpha}.$$

This completes the proof of Lemma 5.

We will now use Lemmas 2 and 5 to obtain the desired estimates on the L^p norms of I_3 and I_4. Applying Lemma 2 to $a = \psi * G_\alpha$, we see, for example, that I_3 is at most a constant independent of ϵ times

$$\int_{|x-y|>\epsilon} \|\psi\|_\infty |f(y)| \left[\frac{\epsilon^{1-\alpha}}{|x-y|^{2-\alpha}} + \frac{1}{|x-y|} \log^+ \left(2 - \frac{2\epsilon}{|x-y|}\right)^{-1}\right] dy$$

$$\leqslant \|\psi\|_\infty \left[\frac{1}{\epsilon} \int_{|x-y|>\epsilon} |f(y)| \frac{\epsilon^{2-\alpha}}{|x-y|^{2-\alpha}} dy \right.$$

$$\left. + \frac{1}{\epsilon} \int_{\epsilon<|x-y|<2\epsilon} |f(y)| \log \left(2 - \frac{2\epsilon}{|x-y|}\right)^{-1} dy\right].$$

Each of the last two integrals is the convolution of $|f|$ with an L^1-approximation to the identity. Hence its L^p norm is at most a constant independent of ϵ times $\|f\|_p$.

Using Lemma 5(ii), we obtain the same estimate for I_4.

Section 3

We now turn to the integrals I_1 and I_2 in (1.4). We consider them together rather than separately;

$$I_1 + I_2 = -\int_{|x-y|>\epsilon} \frac{a(x) - a(y)}{|x-y|^{1+\alpha}} [(Hf)(y) - b \operatorname{sign}(x-y) f(y)]\, dy. \tag{3.1}$$

Let

$$f_+(z) = \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt \qquad \text{for} \quad I(z) > 0$$

and

$$f_-(z) = \int_{-\infty}^{\infty} \frac{f(t)}{t-z}\, dt \qquad \text{for} \quad I(z) < 0.$$

Therefore f_+ is analytic in $I(z) > 0$ with boundary values $f_+(x) = -(Hf)(x) + i\pi f(x)$, and f_- is analytic in $I(z) < 0$ with boundary values $f_-(x) = -(Hf)(x) - i\pi f(x)$. In particular, we obtain a decomposition $f = (f_+ - f_-)/i\pi$, and will consider (3.1) in the cases $f = f_+$ and $f = f_-$. Since $Hg = \text{p.v.}(g * x^{-1})$, it has Fourier transform $(Hg)\hat{} = -i\pi \operatorname{sign} x\hat{g}$. Consequently, $H(f_+) = -i\pi f_+$ and $H(f_-) = i\pi f_-$, and (3.1) for f_+ is

$$\int_{|x-y|>\epsilon} \frac{a(x)-a(y)}{|x-y|^{1+\alpha}} k(x-y) f_+(y)\, dy, \tag{3.2}$$

where $k(x) = i\pi + b$ for $x > 0$ and $k(x) = i\pi - b$ for $x < 0$. Hence, up to multiplication by a constant,

$$k(x) = \begin{cases} 1 & \text{for} \quad x > 0, \\ e^{i\pi(1+\alpha)} = (i\pi - b)/(i\pi + b) & \text{for} \quad x < 0. \end{cases} \tag{3.3}$$

Let $z^{1+\alpha} = e^{(1+\alpha)(\log|z|+i\arg z)} = |z|^{1+\alpha} e^{i(1+\alpha)\arg z}$ be analytic except on the positive imaginary axis and be real on the positive real axis—that is, $-(3\pi/2) < \arg z < (\pi/2)$. With this determination and formula (3.3) for k, we see that (3.2) equals

$$\int_{|x-y|>\epsilon} \frac{a(x)-a(y)}{(x-y)^{1+\alpha}} f_+(y)\, dy. \tag{3.4}$$

We will first compare (3.4) with

$$\int_{-\infty}^{\infty} \frac{a(x)-a(y)}{(x-y-i\epsilon)^{1+\alpha}} f_+(y)\, dy. \tag{3.5}$$

If $|x-y| < \epsilon$,

$$\begin{aligned} \left| \frac{1}{(x-y-i\epsilon)^{1+\alpha}} \right| &= \frac{1}{[(x-y)^2+\epsilon^2]^{(\alpha+1)/2}} \\ &= 0\left(\frac{\epsilon}{[(x-y)^2+\epsilon^2]^{(\alpha+2)/2}} \right). \end{aligned}$$

If, on the contrary, $|x-y| > \epsilon$ then an elementary integration gives

$$\begin{aligned} \left| \frac{1}{(x-y-i\epsilon)^{1+\alpha}} - \frac{1}{(x-y)^{1+\alpha}} \right| &= 0\left(\frac{\epsilon}{|x-y|^{\alpha+2}} \right) \\ &= 0\left(\frac{\epsilon}{[(x-y)^2+\epsilon^2]^{(\alpha+2)/2}} \right). \end{aligned}$$

Hence, by Lemma 2, the difference between (3.4) and (3.5) is less than a constant independent of ϵ times

$$\|\psi\|_\infty \int_{-\infty}^{\infty} |f_+(y)| \frac{\epsilon\, |x-y|^\alpha}{[(x-y)^2+\epsilon^2]^{(\alpha+2)/2}}\, dy.$$

As usual, the last integral is the convolution of $|f_+|$ with an L^1-approximation to the identity. Hence its L^p norm is less than a constant times $\|f\|_p$.

It follows that we may consider the L^p norm of (3.5). To do this we will need several new lemmas and an application of Lemma 3. We begin by observing that if $h(x)$ is infinitely differentiable and has compact support then

$$|h_+^{(\alpha)}(z)| \leqslant \frac{c_\alpha}{(1+|z|^2)^{(\alpha+1)/2}} \qquad (I(z) > 0). \tag{3.6}$$

When $\alpha = 0$, this is clear; h_+ is bounded since h and its Hilbert transform are bounded, and $h_+(z) = O(|z|^{-1})$ as $|z| \to \infty$ since h has compact support. For $\alpha = 1, 2, \ldots$, (3.6) follows by differentiation. For $0 < \alpha < 1$, say,

$$\begin{aligned}
|h_+^{(\alpha)}(z)| = |h_+^{(\alpha)}(x+iy)| &= \left| \int_0^\infty \frac{\partial h}{\partial y}(x, y+s)\, s^{-\alpha}\, ds \right| \\
&\leqslant C \int_0^\infty \frac{s^{-\alpha}\, ds}{1+x^2+y^2+s^2} = 0\left(\frac{1}{1+x^2+y^2}\right)^{(\alpha+1)/2}.
\end{aligned}$$

LEMMA 6. *If h is infinitely differentiable and has compact support and $\epsilon > 0$, then*

$$\int_{-\infty}^{\infty} \frac{h_+(y)}{(x-y-i\epsilon)^{1+\alpha}}\, dy = 0.$$

Proof. The function $h_+(z)(x-z-i\epsilon)^{-1-\alpha}$ is analytic for $I(z) > 0$ and continuous for $I(z) \geqslant 0$. Hence its integral along the boundary of the semidisc $|z| \leqslant R$, $I(z) \geqslant 0$ is zero. But since h_+ is bounded and $\alpha > 0$, the part of the integral over $|z| = R$ tends to zero when $R \to \infty$.

LEMMA 7. *If h is infinitely differentiable and has compact support and $\epsilon > 0$,*

$$\int_{-\infty}^{\infty} \frac{h(x)}{(x-y-i\epsilon)^{1+\alpha}}\, dx = C_\alpha h_+^{(\alpha)}(y+i\epsilon).$$

Proof. Since $(z-y-i\epsilon)^{1+\alpha}$ is analytic for $I(z) < 0$, an argument like that used in Lemma 6 shows that

$$\int_{-\infty}^{\infty} \frac{h_-(x)}{(x-y-i\epsilon)^{1+\alpha}}\, dx = 0.$$

Integrating by parts, we see that

$$\int_{-\infty}^{\infty} \frac{h_+(x)}{(x - y - i\epsilon)^{1+\alpha}}\, dx = \alpha^{-1} \int_{-\infty}^{\infty} \frac{h_+'(x)}{(x - y - i\epsilon)^{\alpha}}\, dx.$$

The function $(z - y - i\epsilon)^\alpha$ is analytic if $R(z) \neq y$ or if $I(z) < \epsilon$. Hence

$$\int_\Gamma \frac{h_+'(z)}{(z - y - i\epsilon)^\alpha}\, dz = 0,$$

where Γ is the boundary of the domain formed by intersecting the semidisc $|z - y| < R$, $I(z) > 0$, the exterior of the disc $|z - y - i\epsilon| \leqslant \eta$, and the exterior of the strip $|R(z) - y| \leqslant (\eta/2)$, $I(z) > \epsilon$.

By (3.6). the part of the integration on $|z| = R$ tends to zero as $R \to \infty$. The part on $|z - y - i\epsilon| = \eta$ is on the order of $\eta^{1-\alpha}$, and therefore tends to zero with η ($0 < \alpha < 1$ as always). A simple limit argument shows that the integrals over the vertical pieces combine in the limit to give

$$\int_0^\infty h_+'(y + i\epsilon + is)\, s^{-\alpha}(e^{i\alpha(3\pi/2)} - e^{-i\alpha(\pi/2)})\, d(is)$$

$$= C_\alpha \int_0^\infty g_+'(y + i\epsilon + is)\, s^{-\alpha}\, ds.$$

This completes the proof of Lemma 7.

We now return to (3.5). Multiplying (3.5) by an infinitely differentiable function $g(x)$ with compact support and integrating, we obtain

$$\int_{-\infty}^{\infty} g(x)\, dx \int_{-\infty}^{\infty} \frac{a(x) - a(y)}{(x - y - i\epsilon)^{1+\alpha}} f_+(y)\, dy$$

$$= - \int_{-\infty}^{\infty} g(x)\, dx \int_{-\infty}^{\infty} \frac{a(y) f_+(y)}{(x - y - i\epsilon)^{1+\alpha}}\, dy$$

$$= C_\alpha \int_{-\infty}^{\infty} a(y) f_+(y)\, g_+^{(\alpha)}(y + i\epsilon)\, dy,$$

by Lemmas 6 and 7 and Fubini's Theorem.

Since $a = \psi * G_\alpha$ is the Bessel potential of ψ, it is also a multiple of a Riesz potential:

$$a(y) = C \int_{-\infty}^{\infty} \frac{\phi(t)}{|y - t|^{1-\alpha}}\, dt,$$

where $\phi = \psi * d\mu$ and $d\mu$ is the finite measure defined by $|t|^{\alpha} = (1 + t^2)^{\alpha/2} d\hat{\mu}$. (See [7].) Substituting this in the integral above, we obtain

$$C_\alpha \int_{-\infty}^{\infty} f_+(y) g_+^{(\alpha)}(y + i\epsilon) \, dy \int_{-\infty}^{\infty} \frac{\phi(t)}{|y - t|^{1-\alpha}} \, dt.$$

In order to be able to interchange integrations, we observe that $\phi \in L^1 \cap L^\infty$. Therefore

$$\int_{-\infty}^{\infty} \frac{|\phi(t)|}{|y - t|^{1-\alpha}} \, dt \leqslant \int_{|y-t|>1} |\phi(t)| \, dt + \int_{|y-t|<1} \frac{\|\phi\|_\infty}{|y - t|^{1-\alpha}} \, dt$$

is bounded. Hence we obtain

$$C_\alpha \int_{-\infty}^{\infty} \phi(t) \, dt \int_{-\infty}^{\infty} f_+(y) g_+^{(\alpha)}(y + i\epsilon) \frac{dy}{|t - y|^{1-\alpha}}$$

$$= C_\alpha \int_{-\infty}^{\infty} \phi(t) \, dt \int_0^{\infty} f_+(t + is) g_+^{(\alpha)}(t + i\epsilon + is) \, s^{\alpha-1} ds. \tag{3.7}$$

We justify the last step by claiming that $H(z) = f_+(z) g_+^{(\alpha)}(z + i\epsilon)$ is the Poisson integral of its boundary values $f_+(y) g_+^{(\alpha)}(y + i\epsilon)$. This is a simple matter, however, since H is analytic for $I(z) > 0$ and $|H(z)| \leqslant C(1 + |z|^2)^{-(\alpha+2)/2}$ by (3.6). Thus H belongs to H^p for all $p \geqslant 1$, and is the Poisson integral of its boundary values. The last formula then follows as in [8, p. 57].

Let us now consider

$$I_\alpha(H)(z) = \int_0^{\infty} H(z + is) \, s^{\alpha-1} ds, \; z = t + iy, \; y > 0,$$

the fractional integral of H of order α. It is not hard to see that $I_\alpha(H)$ belongs to H^p for all $p \geqslant 1$. Indeed,

$$|I_\alpha(H)(z)| \leqslant C \int_0^{\infty} \frac{s^{\alpha-1} \, ds}{(1 + t^2 + y^2 + s^2)^{(\alpha+2)/2}},$$

$$\leqslant C \int_0^{\infty} \frac{s^{\alpha-1} \, ds}{(1 + t^2 + s^2)^{(\alpha+2)/2}} = \frac{C}{1 + t^2}.$$

Therefore, for $p \geqslant 1$,

$$\int_{-\infty}^{\infty} |I_\alpha(H)(t + iy)|^p \, dt \leqslant C \int_{-\infty}^{\infty} \frac{dt}{(1 + t^2)^p} < +\infty.$$

Moreover, $I_\alpha(H)(t + iy)$ has boundary values

$$I_\alpha(H)(t) = \int_0^\infty f_+(t + is)\, g_+^{(\alpha)}(t + i\epsilon + is)\, s^{\alpha-1}\, ds.$$

Using this expression we may rewrite (3.7) as

$$\int_{-\infty}^{\infty} \phi(t)\, I_\alpha(H)(t)\, dt.$$

We use Lemma 3 to majorize this by

$$\begin{aligned} \| \phi \|_\infty \| I_\alpha(H) \|_1 &\leqslant C \| \phi \|_\infty \| S_\alpha(I_\alpha(H)) \|_1 \\ &\leqslant C \| \mu \| \| \psi \|_\infty \left\| \left(\int_{\Gamma(x)} \int y^{2(\alpha-1)} \mid H(t, y)|^2\, dt\, dy \right)^{1/2} \right\|_1, \end{aligned}$$

where $\Gamma(x)$ is a cone with vertex x of uniform shape. If $(Mf)(x) = \max_{\Gamma(x)} |f_+|$, we have the well-known result $\| Mf \|_p \leqslant C \| f \|_p$. (See [10, p. 278].) Thus, with $(1/p) + (1/p') = 1$,

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \phi(t)\, I_\alpha(H)(t)\, dt \right| &\leqslant C \| \psi \|_\infty \| Mf \|_p \| S_\alpha(g_+(\cdot, \epsilon)) \|_{p'}, \\ &\leqslant C \| \psi \|_\infty \| f \|_p \| g_+(\cdot, \epsilon) \|_{p'}, \\ &\leqslant C \| \psi \|_\infty \| f \|_p \| g \|_{p'}, \end{aligned}$$

where the constant is independent of ϵ.

The argument for f_- is similar and we omit it.

SECTION 4

We have shown above that $\| \tilde{F}_\epsilon \|_p \leqslant C \| f \|_p \| \psi \|_\infty$ with C independent of ϵ, provided $F = (aH - Ha)f$, $a = \psi * G_\alpha$, $\psi \in L^1 \cap L^\infty$, and f is infinitely differentiable with compact support. The proof of Theorem 1 will be complete if we show the same is true for $f \in L^p$ and $\psi \in L^\infty$.

Assuming still that $\psi \in L^1 \cap L^\infty$, we can remove the restriction on f as follows. Choose smooth f_m with $\| f - f_m \|_p \to 0$. Then $\| f_m \|_p \to \| f \|_p$, $F_m = (aH - Ha) f_m$ converges in L^p to F and

$$\int_{|h| > \epsilon} \frac{F_m(x + h) - F_m(x)}{| h |^{1+\alpha}}\, dh \to \tilde{F}_\epsilon(x) = \int_{|h| > \epsilon} \frac{F(x + h) - F(x)}{| h |^{1+\alpha}}\, dh$$

in L^p. Therefore since the L^p norm of the expression on the left is less than $C \| f_m \|_p \| \psi \|_\infty$ and also converges to the L^p norm of $\tilde{F}_\epsilon$, we obtain $\| \tilde{F}_\epsilon \|_p < C \| f \|_p \| \psi \|_\infty$.

To remove the restriction on ψ, let γ_m be an approximation to the identity and $\phi_m = \psi * \gamma_m$. Then $\phi_m \to \psi$ almost everywhere and $\| \phi_m \|_\infty \leqslant \| \psi \|_\infty$. Let $\psi_m = \phi_m(x)$ for $| x | \leqslant m$ and $\psi_m = 0$ otherwise. Then $\psi_m \in L^1 \cap L^\infty$, $\| \psi_m \|_\infty \leqslant \| \psi \|_\infty$ and $\psi_m \to \psi$ almost everywhere.

By the Lebesgue dominated convergence theorem, $a_m = \psi_m * G_\alpha \to a = \psi * G_\alpha$ at every point and $\| a_m \|_\infty \leqslant \| \psi_m \|_\infty \leqslant \| \psi \|_\infty$. If $F_m = (a_m H - H a_m) f$ then

$$\left\| \int_{|h| > \epsilon} \frac{F_m(x+h) - F_m(x)}{| h |^{1+\alpha}} \, dh \right\|_p \leqslant C \| f \|_p \| \psi_m \|_\infty \leqslant C \| f \|_p \| \psi \|_\infty ,$$

and it will be enough as usual to show F_m converges to F in L^p. Note $a_m Hf - aHf \to 0$ pointwise and

$$| a_m Hf - aHf |^p \leqslant (\| a_m \|_\infty + \| a \|_\infty)^p \, | Hf |^p \leqslant C \, | Hf |^p \in L^1.$$

Hence $a_m Hf$ converges to aHf in L^p. Finally, the L^p norm of $Ha_m f - Haf$ is less than a constant times $\| a_m f - af \|_p$, which tends to zero by the Lebesgue dominated convergence theorem again.

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